

A length-scale formula for confined Quasi-2D plasmas

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Abstract

Typically a magnetohydrodynamical model for neutral plasmas must take into account both the ionic and the electron fluids and their interaction. However, at short time scales, the ionic fluid appears to be stationary compared to the electron fluid. On these scales, we need consider only the electron motion and associated field dynamics, and a single fluid model called the electron magnetohydrodynamical (EMH) model which treats the ionic fluid as a uniform neutralising background applies. Using Maxwell's equations, the vorticity of the electron fluid and the magnetic field can be combined to give a generalised vorticity field, and one can show that Euler's equations govern its behavior. When the vorticity is concentrated into slender, periodic, and nearly parallel (but slightly 3D) filaments, one can also show that Euler's equations simplify into a Hamiltonian system and treat the system in statistical equilibrium, where the

filaments act as interacting particles. In this paper, we show that, under a mean-field approximation, as the number of filaments becomes infinite (with appropriate scaling to keep the vorticity constant) and given that angular momentum is conserved, the statistical length scale, R , of this system in the Gibbs canonical ensemble follows an explicit formula, which we derive. This formula shows how the most critical statistic of an electron plasma of this type, its size, varies with angular momentum, kinetic energy, and filament elasticity (a measure of the interior structure of each filament) and in particular it shows how 3D effects cause significant increases in the system size from a perfectly parallel, 2-D, one-component Coulomb gas.

1 Introduction

Plasma confinement is critical to magnetic nuclear fusion. Given a neutral plasma in a strongly confining magnetic field, columns of electrons and ions align, and the plasma behaves as a two-dimensional fluid. However, one of the great obstacles to confinement comes from the increasing impact of 3D effects, defects in the alignment of columns due to fluctuations, as the plasma is further confined. Two-dimensional models are insufficient to deal with this problem, and, in statistical equilibrium, fail to accurately predict the length scale of dense plasmas in the unbounded plane under conservation of angular momentum.

Because of its mathematical simplicity and apparent applicability to highly aligned magnetically confined plasmas (and other Euler-fluids), two-dimensional

fluid models such as Onsager’s Point Vortex Gas [28] have dominated the statistical equilibrium approach to plasmas and other fluids for decades. The literature on the point vortex gas and its relatives is quite extensive. Seminal papers such as Onsager’s and Joyce and Montgomery [16], Edwards and Taylor [10] describe negative temperature states in fluids and guiding centre plasmas (where temperature is virtual, treating vortices as if they were molecules rather than collections of them). Mathematical rigour has been applied in the form of existence proofs for the asymptotic mean-field approach to this 2D problem in Caglioti et al. [5], Kiessling [17], Kiessling and Spohn [18]. (For a more complete overview see Lim [23], Lim and Nebus [25].) Two-dimensional models work well for plasmas that are not well confined (sparse), but they ignore the internal entropy of the filaments—their “unwillingness” to remain aligned even under strong confinement. This entropy creates 3D effects that become noticeable when the confining field brings filaments close together.

Theoretical derivations for three dimensional fluids are difficult and tend to focus on single columns [14, 6] and [21, 29]. In 1995, Uby et al. [30] described the dynamics of the combined magnetic and vorticity field (known as a generalised vorticity field) of a single filament of electrons in a neutral plasma using the short time scale electron magnetohydrodynamical (EMH) model. Earlier Kinney et al. [20] had used this same model to describe the interaction of a system of perfectly parallel (2D) filaments of generalised vorticity (an alternative to the guiding centre 2D approach to plasmas but mathematically equivalent), suggesting that the two approaches could be combined into a system of interacting, slightly 3D,

nearly parallel filaments of generalised vorticity such as Klein et al. [22] has done for Navier-Stokes fluids. At the same time, several authors including E [9] have considered the dynamics of vorticity in the context of superconductivity and superfluids.

This nearly parallel electron vorticity model provides a platform from which to investigate the relatively unknown impact of 3D effects on large scale, short duration structures in confined plasmas.

The EMH model is a simplification of the two-fluid magnetohydrodynamical model. As time intervals become small, the ionic fluid appears stationary and its behavior can be treated as a neutralizing background. In this model, the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, and the charged fluid vorticity, $\omega = \nabla \times \mathbf{v}$, combine into a general vorticity field $\Omega = \nabla \times \mathbf{p}$ where the generalised momentum, $\mathbf{p} = m\mathbf{v} - e\mathbf{A}$, m is the electron mass, $-e$ is the electron charge, \mathbf{v} is the fluid velocity field, and \mathbf{A} is the magnetic vector potential field. For a brief overview of the model, see Uby et al. [30]. A detailed model discussion can be found in Gordeev et al. [12].

In the case of dense turbulent plasmas, a statistical approach is preferable to direct numerical PDE simulations because of the large number of dimensions (in the thousands). Given that we assume that the generalised vorticity field takes the form of nearly parallel slender filaments with appropriate asymptotic constraints (described below), the EMH model describes a Hamiltonian system. Even though the model is for short time scales, this Hamiltonian system can be studied in a statistical equilibrium ensemble with the caveat that the

macroscopic coherent structures self-organized over a relaxation time-scale that is short compared to the dissipation time-scale.

For an analogous example, consider the large vortex that forms from initially swirling tea in a cup - a case of freely decaying quasi-2D turbulence [27]. While this vorticity does not last long, it can be properly treated as the statistical equilibrium result [8] where the time-scale for the inverse cascade of energy to large spatial scales is much shorter than the Ekman dissipation time-scale. Moreover, over short durations energy dissipation in the interior of the flow can be ignored because the viscous dissipation time-scale is much longer than the Ekman dissipation time-scale. Of course, without energy dissipation, either in the interior of the flow or in the Ekman boundary layers, the tea would swirl forever. A more rigorous approach is discussed in [13] where it is shown that systems not in thermal equilibrium and irreversible systems can often undergo symmetry-breaking phase transitions and self-organize like their counterparts in equilibrium.

The same is true for confined plasmas, and we are particularly interested in short duration structures and how they interfere with attaining sufficient confinement for fusion. What remains is to choose an appropriate statistical model.

Two statistical equilibrium ensembles are applicable to this problem: (1) the canonical in which we assume that the system is not in isolation and that energy can be exchanged with an exterior medium and (2) the microcanonical where the system is isolated and energy and angular momentum are absolutely

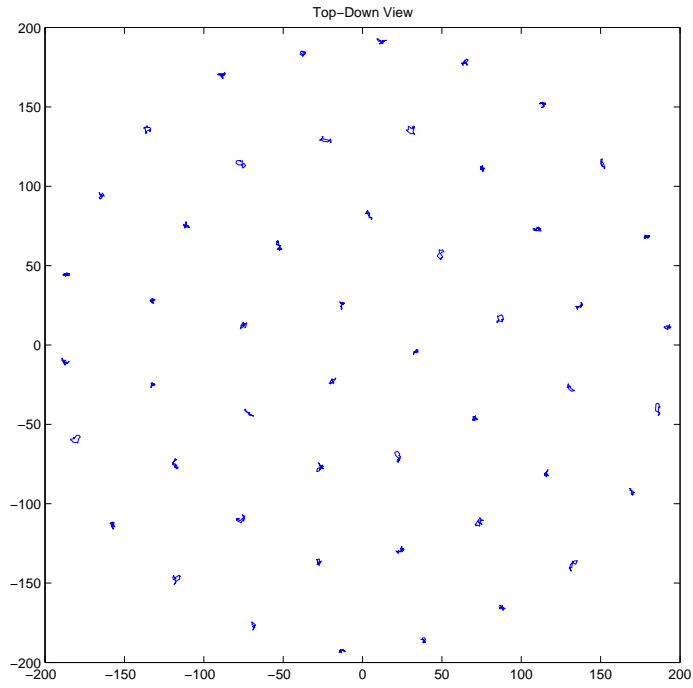


Figure 1: Shown here in top-down projection, nearly parallel vortex filaments, at low density and high strength of interaction, are well-ordered into a 2-D triangular lattice known as the Abrikosov lattice from type-II superconductors ([1]). This figure (taken from a single Monte Carlo sample) shows how the quasi-2D model is essentially a 2-D model for these parameters.

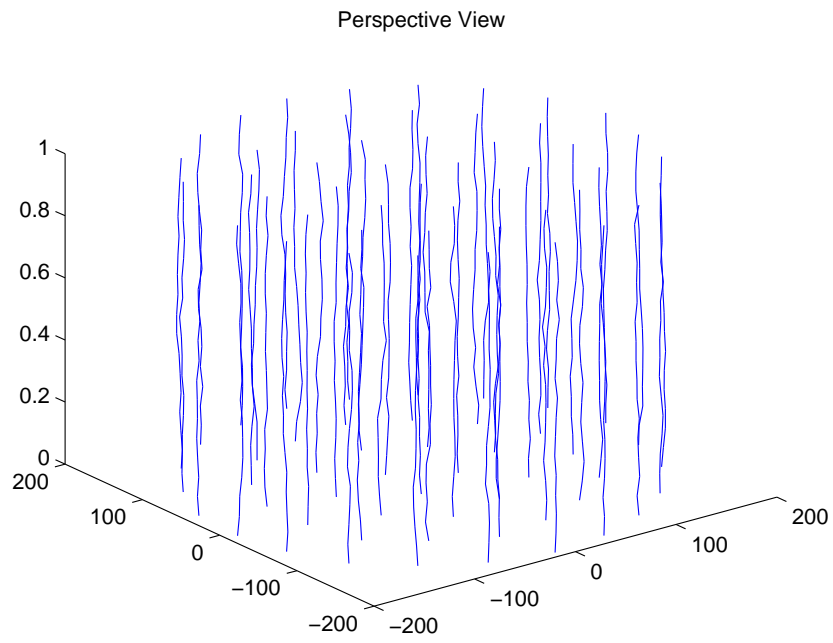


Figure 2: Nearly parallel vortex filaments are nearly parallel to the z -axis in an asymptotic sense that their deviation from straight is a small parameter. They are infinite in length but have a period L . Here there are 50 filaments.

conserved. The microcanonical ensemble would model a plasma experiences no outside influence. In a tokamak, this would imply that the magnetic confinement is shut off. The canonical ensemble describes the case where energy is allowed to pass between the plasma and the magnetic field (effectively an infinite reservoir). In a previous paper we considered the microcanonical ensemble [2]. Here we address the problem of when there is an energy/angular momentum exchange medium surrounding the system, i.e. the canonical ensemble.

Because of the difficulty in calculating partition functions (and thus solving the ensemble exactly), we focus on calculating an individual moment. The primary moment for any system that is translation invariant and on the unbounded plane is the second, essentially size. The size of the system gives us the most intuitive idea of how the system is behaving, and, for confined plasmas, it gives us a good idea of how easy it is to squeeze the plasma magnetically. Lim and Assad [24] has calculated the second moment of the canonical guiding centre plasma, giving an explicit formula for low temperature in the mean-field limit. The formula for the nearly parallel system in the same limits (if it exists in explicit form) has never been calculated.

In this paper, we calculate a novel explicit formula for the size (second moment) of the system of nearly parallel generalised vorticity filament with conserved angular momentum. This formula comes from a free energy minimisation derived from the Gibbs canonical ensemble using the rigorous mean-field theory of Lions and Majda [26]. We validate this formula with Path Integral Monte Carlo simulations. Our formula and Monte Carlo results agree well and

both indicate that 3D effects become significant at a particular density. We show that at high-densities point vortices and nearly parallel vortex filaments have qualitatively different behavior. While the size of the point vortex plasma collapses with increasing temperature [24], nearly parallel filaments, which at low density behave exactly like point vortices, reverse their collapse, causing expansion (Sec. 6). We hypothesise that this reversal arises from the additional entropy of the system present in individual filaments.

2 Background

2.1 Equations of Motion

We derive the standard electron magnetohydrodynamical (EMH) equations of motion in the manner of Uby et al. [30].

For an electron fluid (neglecting collisions with the neutralising background),

$$m \left[\frac{\partial \mathbf{v}}{\partial t} \right] = \frac{-\nabla P}{n} - e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (1)$$

where m and $-e$ are the electron mass and charge, \mathbf{v} , n , and P are electron fluid velocity, number density, and pressure. When electron displacement current is neglected, Ampère's law for the magnetic field gives

$$\nabla \times \mathbf{B} = -\mu_0 e n \mathbf{v}, \quad (2)$$

and Faraday's law determines the electric field \mathbf{E} .

Given that the pressure is a function of number density and the fluid incom-

pressible, taking the curl of 1,

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\Omega}), \quad (3)$$

where $\boldsymbol{\Omega} = \nabla \times \mathbf{p}$ is generalized vorticity and $\mathbf{p} = m\mathbf{v} - e\mathbf{A}$, generalized momentum.

Combining 2 with the frozen field equation, 3, we have

$$\boldsymbol{\Omega} = -e[\mathbf{B} + \nabla \times (\lambda^2 \nabla \times \mathbf{B})], \quad (4)$$

where $\lambda = c/\omega_{pe}$ is the penetration depth. The electron plasma frequency is $\omega_{pe}^2 = ne^2/m\varepsilon_0$.

For a homogeneous plasma,

$$\boldsymbol{\Omega} = \mathbf{B} - \lambda^2 \nabla^2 \mathbf{B}, \quad (5)$$

where the magnetic field and generalized vorticity have been scaled $-e\mathbf{B}/m \rightarrow \mathbf{B}$ and $\boldsymbol{\Omega}/m \rightarrow \boldsymbol{\Omega}$. Equation 2 becomes $\mathbf{v} = \lambda^2 \nabla \times \mathbf{B}$ and along with 3 and 5 describes the motion of the electron fluid.

Equation 3 has the same form as the vorticity equation for ordinary fluids from which Lions and Majda [26] derived the nonlinear Schrödinger PDE and Hamiltonian equations for the system of nearly parallel filaments in this paper.

The stability of vortices with respect to short wave perturbations is shown in Ivonin [15].

2.2 Gibbs Canonical Ensemble

In formal notation, a canonical Gibbs' measure has that if a state s has energy E_s and angular momentum M_s , then the probability of s ,

$$P(E_s, M_s) \propto e^{-\beta E_s - \mu M_s}, \quad (6)$$

where β and μ are constants. Since $\sum_s P(E_s, M_s) = 1$, $P(E_s, M_s) = Z^{-1} e^{-\beta E_s - \mu M_s}$, where $Z = \sum_s e^{-\beta E_s - \mu M_s}$ [8]. Alternatively, we can let $p = \mu/\beta$ constant (for plasmas, the Larmor frequency) and enthalpy be $H_s = E_s + pM_s$ in which case we have an ensemble in a single conserved quantity: $P(H_s) = Z^{-1} \exp(-\beta H_s)$.

For historical reasons β is known as inverse “temperature” although it is not directly related to its molecular equivalent because the “molecules” here are vortices. The multiplier μ reflects the relationship between the angular momentum and the energy and μ/β is the Larmor frequency (ω_{Larmor}) [24] and can be thought of as a “pressure” term defining how squeezed the system is. Because β has units of inverse energy and μ has units of frequency over energy and μ/β inverse time or frequency.

There is little in the literature to say what β is or how to measure it although it does have a well-defined meaning. The temperature of a microcanonical system is well-defined as, $1/T_{mc} = dS(E, N)/dE$, the rate of increase of entropy with energy where the microcanonical entropy $S(E, N)$ is given by the natural log of the microcanonical partition function $Z_{mc}(E, N)$. One way to calculate the canonical inverse temperature β is to invoke a large deviation result that holds even in case the Gibbs (canonical) measure is weakly inequivalent to the

microcanonical ensemble, that is, the Gibbs measure or canonical partition function $Z_c(\beta, N)$ is overwhelmingly concentrated at the most probable macrostate, for which the relationship $\beta = dS(E, N)/dE$ holds. In this case, the definitions of temperature in the canonical and microcanonical formulations are in agreement, even though the two ensembles are not strictly equivalent - for example, this weak inequivalence allows the specific heat in the microcanonical ensemble to take on negative values whilst its canonical counterpart must be positive. This extends to weakly inequivalent ensembles, the standard inverse procedure for determining canonical temperature as a function of the fixed energy level E in the microcanonical ensemble through the relationship $\langle E \rangle_c(T(E)) = E$ where the subscript c denotes the canonical average.

In the EMH model, β is precisely related to the kinetic energies associated with the macroscopic motions (flows) and magnetic fields in the plasma, and thus to the total generalised vorticity, $\int d^3x \Omega(x, y, z)$.

2.3 Length Scale for Point Vortex Equilibrium Statistics

The low-temperature (asymptotic as $\beta \rightarrow \infty$) statistics of strictly-2D point vortices, guiding centre plasmas, and 2D EMH generalised vorticity in the unbounded plane with angular momentum conserved are well understood [23]. The Point-Vortex Gas Model has the energy functional,

$$E_N^{2D} = - \sum_{j>i} \lambda_i \lambda_j \log |\Psi_i - \Psi_j|^2, \quad (7)$$

where λ_i and λ_j are the circulation constants for point vortices i and j and Ψ_i and Ψ_j are their positions in the complex plane. This Hamiltonian derives from

the 2D Euler equations in vorticity form given that ω , the field, has the form

$$\omega(\Psi) = - \sum_j \lambda_j \delta(\Psi - \Psi_j),$$

where Ψ is a position in the complex plane. [8].

In their paper, Lim and Assad [24] show variationally that the mean square vortex position (variance) for 2D point vortices,

$$R_{2D}^2 = \langle N^{-1} \sum_i |\Psi_i|^2 \rangle, \tag{8}$$

has a formula as $N \rightarrow \infty$. If $\Omega = \sum_i \lambda_i$ is the total circulation and we let $\lambda_i = \Omega/N$, then taking the limit on the Gibbs canonical ensemble, we have the non-dimensional variance,

$$R_{2D}^2 = \frac{\Omega\beta}{4\mu} \tag{9}$$

In Monte Carlo simulations of an ensemble of 1000 point vortices, Assad and Lim [3] showed that the distribution of vortices is almost uniform and axisymmetric, meaning that the probability distribution of vortices is nearly a perfect cylinder, suggesting that R_{2D}^2 is the only valuable statistic.

3 The Nearly Parallel Vortex Filament Model

We define nearly parallel vortex filaments as in [22]:

Nearly parallel vortex filaments are $C^2([0, L])$ curves that we can represent with a complex parametrisation $\Psi_i(\tau)$ where $\Psi_i(\tau) = x_i(\tau) + iy_i(\tau)$ and $\tau \in [0, L]$. They have a special asymptotic form, Given that $L \in O(1)$, if we take any two values of τ , τ_0 and τ_1 such that $\tau_0 < \tau_1$, and let $\Delta\tau = \tau_1 - \tau_0$ such

that $\Delta\tau \in O(\varepsilon)$ where $\varepsilon \ll 1$, then for any filament i , the amplitude $|\Psi_i(\tau_1) - \Psi_i(\tau_0)| \in O(\varepsilon^2)$. In words this means that, for a small rise of length ε in the filament, the amplitude must be on the order of ε^2 . This assumption guarantees a certain degree of straightness in the filament that allows for the derivation of the quasi-2D equations of motion. The other asymptotic assumption is of the vortex core-size, h , which has the property $h \ll \varepsilon$. In this model we assume that the vortices have no cross-section, i.e. the vorticity field, ω , has the form,

$$\omega = \sum_i \delta(\Psi(\tau) - \Psi_i(\tau)), \quad (10)$$

where $\Psi(\tau) = x(\tau) + iy(\tau)$ corresponds to the 3D Cartesian position $(x(\tau), y(\tau), \tau)$.

Combining the energy functionals of Kinney et al. [20] and Uby et al. [30] we have,

$$E_N = \alpha \int_0^L d\tau \sum_{i=1}^N \frac{1}{2} \left| \frac{\partial \Psi_i(\tau)}{\partial \tau} \right|^2 - \int_0^L d\tau \sum_{i=1}^N \sum_{j>i}^N \log |\Psi_i(\tau) - \Psi_j(\tau)|, \quad (11)$$

where α is the core-elasticity constant in units of energy/length and results from an asymptotic matching procedure [30, 21]. This functional resembles the 2D point vortex energy in that the interaction is logarithmic in the plane. For two filaments i and j , only points in the same plane, the same value of τ , interact. Two points at different values of τ only interact if they are both on the same filament. The first term is a local self-induction term that causes the filaments to bend. The additional conserved quantity, angular momentum, is given by,

$$M_N = \sum_i \int_0^L d\tau |\Psi_i(\tau)|^2, \quad (12)$$

with the assumption that the magnetic field \mathbf{B} is uniform and axisymmetric, centred around the central axis. The angular momentum is not generalised, however, because, as is well known, an external magnetic field cannot influence equilibrium statistics. Therefore, M_N is only angular momentum and contains none of the magnetic moment.

Lions and Majda [26] introduce a broken segment model in Section 2.2, Equation 2.20 of their paper. For each i , $\Psi_i(\tau)$ is piecewise linear with M segments. Each vertex or “bead” is at a multiple of $\epsilon = L/M$. Therefore, we define $\psi_i(k) = \Psi_i((k-1)\epsilon)$, and the curve for filament i is given by the vector $\Psi_i = (\psi_i(1), \dots, \psi_i(M))$. With this representation we rewrite the conserved quantities,

$$E_N(M) = \alpha \sum_{i=1}^N \sum_{k=1}^M \frac{1}{2} \frac{|\psi_i(k+1) - \psi_i(k)|^2}{\epsilon} - \sum_{i=1}^N \sum_{j>i}^N \sum_{k=1}^M \epsilon \log |\psi_i(k) - \psi_j(k)|, \quad (13)$$

$$M_N(M) = \sum_{i=1}^N \sum_{k=1}^M \epsilon |\psi_i(k)|^2. \quad (14)$$

The Gibbs probability measure for a state $s = (\Psi_1, \dots, \Psi_N)$ is then

$$P(s) = Z^{-1} \exp(-\beta E_N(M) - \mu M_N(M)), \quad (15)$$

where the partition function has the form,

$$Z_N(M) = \int_{\mathcal{C}^{MN}} d\Psi_1 \cdots d\Psi_N \exp(-\beta E_N(M) - \mu M_N(M)). \quad (16)$$

NB: A careful choice of parameters β , μ , etc. causes any non-nearly parallel states to have such large energies that their probabilities are negligible. For such parameter ranges, we can assume that the Gibbs canonical model is physical.

Our Monte Carlo simulation naturally rejects states with non-nearly-parallel lines.

We note that while it is possible to simulate the system with Monte Carlo it is not possible to solve explicitly for Z_N for any given value of M . In their paper, Lions and Majda [26] go on to derive a non-linear Schroedinger equation that can give approximate values for $P(s)$. Their PDE captures a great deal of the statistics, but they do not provide any explicit formula for the length scale R , which is our goal. We refer the reader to their paper to learn more about their derivation and the model.

4 Free Energy Theory

4.1 Free Energy of Most-Probable Macrostate

Given a functional for the free energy for a system, F , it is possible to solve for the statistics of the most-probable macrostate by minimizing F with respect to the desired statistic. In our case the statistic is defined as:

The mean square vortex position for a given number of filaments N ,

$$R^2 = \left\langle (LN)^{-1} \sum_{i=1}^N \int_0^L d\tau |\Psi_i(\tau)|^2 \right\rangle, \quad (17)$$

where $\Psi_i(\tau) = x_i(\tau) + iy_i(\tau)$ is a complex number representing the 2D position of filament i at $z = \tau$ and the average $\langle * \rangle$ is with respect to the 3D Gibbs probability measure in Ψ_i , $P_N = Z_N^{-1} \int d\Psi_1 \cdots d\Psi_N \exp(-\beta E_N - \mu M_N)$, where E_N is defined by 11, M_N by 12.

In our system we write the free energy as follows:

$$F_N = \langle E_N \rangle + \frac{\mu}{\beta} \langle M_N \rangle - \frac{1}{\beta} S_N, \quad (18)$$

where S_N is the entropy, $\langle E_N \rangle + \frac{\mu}{\beta} \langle M_N \rangle$ is the total enthalpy. Temperature $T = \frac{1}{\beta}$.

The function Z_N can be considered a sum over microstates, s_i , or it can be an average over a set of macrostates, σ_j . Equation 16 is a sum over microstates. Alternatively, the partition function is given by the formula

$$Z_N = \sum_j \exp(-\beta E_N[\sigma_j] - \mu M_N[\sigma_j]) P[\sigma_j], \quad (19)$$

where E_N is the energy, M_N is the angular momentum, and P is the probability for macrostate σ_j .

Since $S_N[\sigma_j] = \log P[\sigma_j]$, using the formula 18, we can say,

$$\begin{aligned} Z_N &= \sum_j \exp(-\beta E_N[\sigma_j] - \mu M_N[\sigma_j] + S[\sigma_j]) \\ &= \sum_j \exp(-\beta F_N[\sigma_j]). \end{aligned} \quad (20)$$

Because of conservation laws, we assume in physics that the most-probable macrostate or energy-state, $j = m$, has a probability so much larger than the probabilities of all other macrostates that sum contributions from other macrostates can be neglected and

$$Z_N = \exp(-\beta F_N[\sigma_m]). \quad (21)$$

Therefore, we have an equation for the free energy,

$$F_N = -\frac{1}{\beta} \log Z_N. \quad (22)$$

Even though this free energy is only the free energy of the most-probable macrostate, it can be considered the system's free energy.

As mentioned in Section 3, we cannot solve for Z_N . We need to approximate it to derive the free energy functional. One way to do this is with a mean-field theory.

4.2 Mean Field Theory

In Lions and Majda [26], they develop a mean-field PDE that describes the probability distribution of the filaments in top-down projection. One of the solutions to that PDE is a uniform distribution which is exact in the limit as $\beta \rightarrow \infty$ (low-temperature) and $N \rightarrow \infty$, given appropriate scaling of other parameters to avoid a blow-up of the energy functional. The uniform distribution is over a cylinder of radius $2R$ and height L . It causes the interaction energy and angular momentum to become constants and imposes a “spherical constraint” on the integral. We include this constraint as a micro-canonical or exact constraint into the partition function,

$$Z'_N(M) = \left\{ \int d\Psi e^{(-\beta E(M) - \mu M R^2)} \delta \left(R^2 - \sum_{k=1}^M M^{-1} |\psi(k)|^2 \right) \right\}^N, \quad (23)$$

where

$$E(M) = \alpha \sum_{k=1}^M \frac{1}{2} \frac{|\psi_i(k+1) - \psi_i(k)|^2}{\epsilon} - N/4 \sum_{k=1}^M \epsilon \log R^2. \quad (24)$$

Note that the sum over N has become a power because the iterated integrals have separated into a product of integrals with the disappearance of the inter-dependencies in the interaction term.

This constraint eliminates fluctuations in interaction energy and angular momentum (as the limit in β of the mean-field theory implies), leaving only fluctuations in local self-induction energy. This elimination provides the best opportunity for comparison with the 2D length scale work of [24] that also eliminates these two types of fluctuation.

4.3 Spherical Model Solution

Equation 23 is a multidimensional Gaussian integral with a spherical constraint—so called because it forces the vector Ψ to stay on the $M - 1$ -sphere of radius \sqrt{MR} . To evaluate it we turn to the spherical model of [4]. The spherical model was first used in fluid mechanics in an energy-ensrophy theory for transitions to super-rotation in barotropic flows coupled to massive rotating spheres [25].

The spherical model comprises a number of steps for evaluating integrals of the form of Equation 23, beginning by putting the Dirac-delta function into integral form and ending with a steepest descent evaluation of the integral over Ψ .

In integral (Fourier) form the Dirac-delta reads:

$$\delta\left(MR^2 - \sum_{k=1}^M |\psi(k)|^2\right) = \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} \exp\left[-i\sigma\left(MR^2 - \sum_{k=1}^M |\psi(k)|^2\right)\right], \quad (25)$$

and allows us to combine the function in the exponent in Equation 23 with the exponent of the spherical constraint. (Whereas before σ was used as a symbol for macrostate, here it is an integration variable.)

4.4 Free Energy Derivation for R

Now we can determine the free energy functional, which ought to be minimal under the constraints of the system, via steepest descent. Determining the free energy as a function of the particular statistic, R , is more useful than having an equation for the partition function itself because we can determine R by minimizing the free energy w.r.t. it and then solving for R .

Given the partition function defined in 23 and the free energy in 18, as $M, N \rightarrow \infty$, the value for R^2 , defined in 4.1, for the minimal free energy is,

$$R^2 = \frac{\beta'^2 \alpha' + \sqrt{\beta'^4 \alpha'^2 + 32\alpha' \beta' \mu}}{8\alpha' \beta' \mu}, \quad (26)$$

where $\alpha' = \alpha/N$ and $\beta' = \beta N$ are scaled non-extensively. The proof of this result is given in the appendix, and we drop primes henceforth.

The resulting expression for the square length scale R^2 is useful for comparison with the length scale result of Lim and Assad [24] because, if we take the limit

$$\lim_{\alpha \rightarrow \infty} \frac{\beta^2 \alpha + \sqrt{\beta^4 \alpha^2 + 32\alpha \beta \mu}}{8\alpha \beta \mu} = \frac{\beta}{4\mu}, \quad (27)$$

we get back the 2D point vortex result for the length scale, which shows that our formula and the Lim and Assad [24] formula agree for perfectly straight filaments.

However, for finite α the two formulae show a significant difference. The 2D formula is linear in β , our formula is non-linear. In fact, for decreasing β , the sign of the slope of our formula changes at $\beta = \beta_0$, where

$$\beta_0^3 = \frac{4\mu}{\alpha}. \quad (28)$$

That the system collapses and then starts to expand as “temperature”, $1/\beta$, increases indicates a significant departure from the strictly-2D where the system size only collapses. In Section 6, we show that that Monte Carlo confirms this result and that the straightness assumptions of the model hold through much of the expansion phase.

5 Monte Carlo

Path Integral Monte Carlo methods emerged from the path integral formulation invented by Dirac that Richard Feynman later expanded [31], in which particles are conceived to follow all paths through space. One of Feynman’s great contributions to the quantum many-body problem was the mapping of path integrals onto a classical system of interacting “polymers” [11]. D. M. Ceperley used Feynman’s convenient piecewise linear formulation to develop his PIMC method which he successfully applied to He-4, generating the well-known lambda transition for the first time in a microscopic particle simulation [7]. Because it describes a system of interacting polymers, PIMC applies to classical systems that have a “polymer”-type description like nearly parallel vortex filaments.

PIMC has several advantages. It is a *continuum* Monte Carlo algorithm, relying on no spatial lattice. Only time (length in the z -direction in the case of vortex filaments) is discretised, and the algorithm makes no assumptions about types of phase transitions or trial wave-functions.

The Monte Carlo simulation begins with a random distribution of filament

end-points in a square of side 10, and there are two possible moves that the algorithm chooses at random:

1. Moves a filament's end-points, $\psi_i(1)$ and $\psi_i(M + 1)$. The index i is chosen at random, and the filament i 's end-points moved a uniform random distance.
2. Keeps end-points stationary and, following the bisection method of Cepereley (Figure 3), grows a new internal configuration for a randomly chosen filament

[7].

In each case, the energy of the new state, s' , is calculated and retained with probability

$$A(s \rightarrow s') = \min \left\{ 1, \exp \left(-\beta [E_N^{s'}(M) - E_N^s(M)] - \mu [M_N^{s'}(M) - M_N^s(M)] \right) \right\}, \quad (29)$$

where s is the previous state. This effectively samples states from the Gibbs probability distribution in Equation 15.

Our stopping criteria is graphical in that we ensure that the cumulative arithmetic mean of the energy,

$$E_{cum}^k = k^{-1} \sum_{i=1}^k E_N(s_i) + \frac{\mu}{\beta} M_N(s_i), \quad (30)$$

where s_i refers to the state resulting from the i th move and k is the current move index, settles to a constant. The energy is almost guaranteed to settle in the case of the Gibbs' measure because of the tendency for the system to

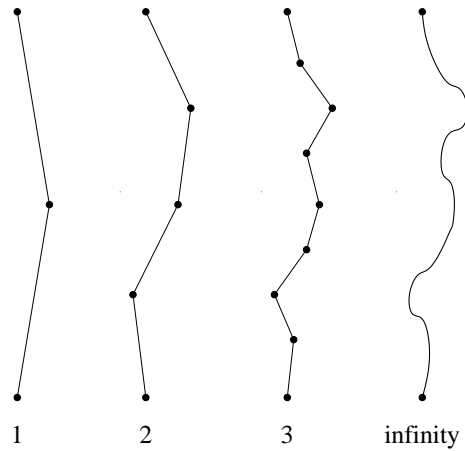


Figure 3: The bisection algorithm works by bisecting the filament to sample point positions. First the center point is selected, then the two points half-way from the center to the end-points, then four more points, eight, and so on until some maximum number of points are sampled. These are snapshots of one filament at different steps in the sampling process.

select a particular energy state (mean energy) and remain close to that state. Typically, we run for 1 million moves (accepted plus rejected) or 50,000 sweeps for 20 vortices. Afterwards, we collect data from about 200,000 moves (1000 sweeps) to generate statistical information.

6 Results

6.1 Comparison

We simulated a collection of $N = 20$ vortices each with a piecewise linear representation with $M = 1024$ segments and ran the system to equilibration, determined by the settling of the mean and variance of the total energy. We ran the system for 20 logarithmically spaced values of β between 0.001 and 1 plus two points, 10 and 100. We set $\alpha = 10^7$ (enforcing straightness), $\mu = 2000$, and $L = 10$. Decreasing β simulates an increase in temperature, $1/\beta$. Units are arbitrary, with those chosen for α and β (e.g. ergs per centimetre and inverse ergs) determining those of R (e.g. centimetres) or vice versa. We calculate two arithmetic averages: the mean square vortex position,

$$R_{MC}^2 = (MN)^{-1} \sum_{i=1}^N \sum_{k=1}^M |\psi_i(k)|^2, \quad (31)$$

and the mean square amplitude per segment,

$$a^2 = (MN)^{-1} \sum_{i=1}^N \sum_{k=1}^M |\psi_i(k) - \psi_i(k+1)|^2, \quad (32)$$

where $\psi_i(M+1) = \psi_i(1)$.

Measures of the Monte Carlo R_{MC}^2 , Equation 31, correspond well to the 3D R^2 , Equation 26, in Figure 4 whereas, the strictly-2D R_{2D}^2 , Equation 9, continues to decline when the others curve up with decreasing β values, suggesting that the 3-D effects are not only real in the Monte Carlo but that the mean-field is a good approximation with these parameters.

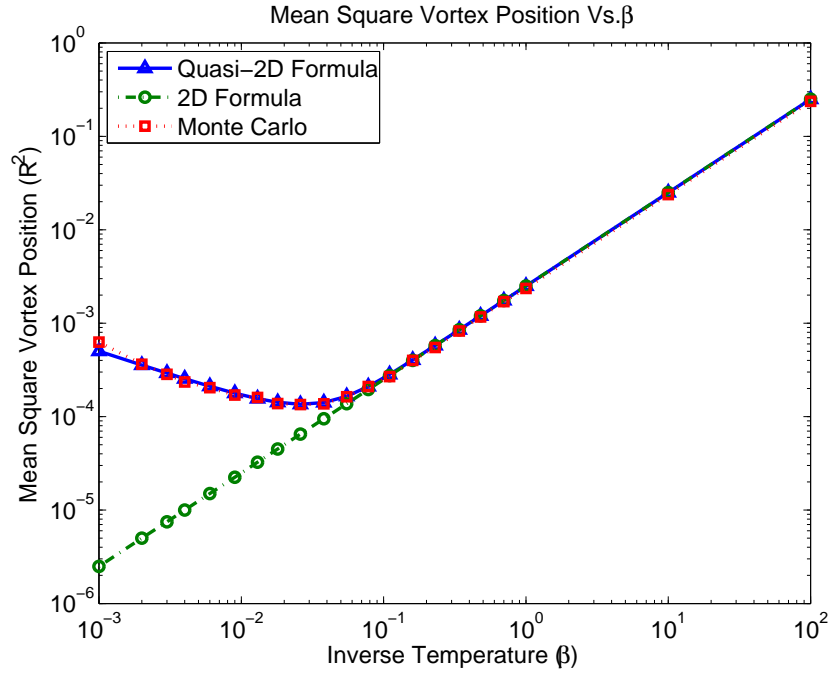


Figure 4: The mean square vortex position, defined in Equation 31, compared with Equations 26 and 9 shows how 3-D effects come into play around $\beta = 0.16$. That, In the 2D formula, R_{2D}^2 continues to decrease while the Monte Carlo and the quasi-2D radii curve upwards with decreasing β suggests that the internal variations of the vortex lines have a significant effect on the probability distribution of vortices.

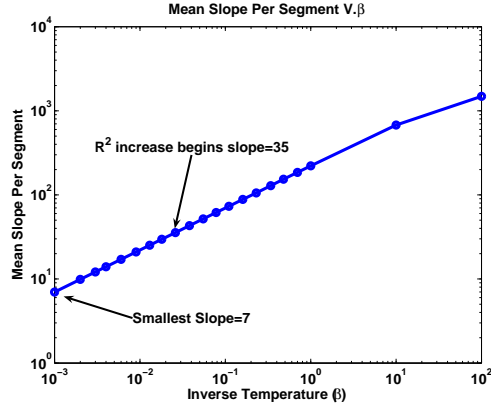


Figure 5: This figure shows the mean slope per segment, ϵ/a , where $\epsilon \sim 0.0098$ and a is given by Equation 32, and that straightness holds for all β values. The point at which R^2 begins to increase with decreasing β (Figure 4) has a mean slope of 35, and, even at the smallest $\beta = 10^{-3}$, the segments have an average angle of 82° with respect to the complex plane.

6.2 Straightness Holds

In order to be considered straight enough, we need

$$a \ll \frac{L}{M} = \frac{10}{1024} \sim 0.0098. \quad (33)$$

Straightness holds for all β values, shown in Figure 5. These conditions hold on average. We ignore extreme low-probability cases as not contributing significantly to the statistics. These straightness constraints, coupled with filaments having no attractive interactions. Hairpins (kinks or, in quantum terminology, instantons) do not occur.

Aside from straightness constraints, the reader might question whether al-

lowing vortex filaments to entangle violates the model's assumptions. While this is a valid concern, it is not an assumption of the model. Since this fluid is almost-everywhere invicid, it contains asymptotically small regions of non-zero viscosity, and, while a totally invicid fluid cannot allow vortices to change topology (cross over each other) from state to state, an almost-everywhere invicid fluid allows vortex reconnections and cross-overs to occur due to microscopic viscous effects. Therefore, in our simulations vortices are allowed to cross one-another. We are not claiming to model vortex reconnection, which is a mysterious process, but only the before and after effects of it.

Concerning the question of how vortices can cross one-another and still remain nearly parallel, we point to the extremely high-density (tiny value of R^2) which allows even the straightest filaments to entangle.

7 Conclusion

The decreasing- β , R -expansion suggests that, by adding degrees of freedom to the 2D model to make it a quasi-2D model, we add a mechanism for the vortices to resist confinement through entropic effects. As β decreases past the transition, the system's R^2 goes from being the result of interaction-versus-angular momentum competition to an entropy-versus-angular momentum competition. In the 2D system this 3rd dimension entropy is not there. Although there is another kind of entropy in the 2D model that will slow the compression as $\beta \rightarrow 0$ to a constant value, it is not enough to cause an increase in the system size. In

the 3D system, the degrees of freedom are exponentially greater, which causes the expansion seen in Figure 4.

The explicit formula for the length scale provides a way to quickly explore the parameter space of the statistical system and find areas of interest for more in-depth study, and to determine hard to measure parameters, such as α and β , from the system size, R . The periodic geometry we are using applies to very large tori as well as columns. Therefore, these results can apply to a tokamak as well as stellar atmospheres. The expansion of the system shows that tight confinement of plasma columns is much more difficult when 3D effects are taken into account than a 2D system implies. However, non-uniformities in the distribution of filaments may allow tighter confinement towards the core than in the periphery of the system. Such a possibility has been suggested by Kiessling and Neukirch [19] and in our earlier paper on the microcanonical ensemble Andersen and Lim [2]. Therefore, further research on the density distribution is called for.

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A Proof of Formula for R^2

The theorem follows from a steepest descent argument. In employing steepest descent, we need to start with an integral of the form, $\int dx e^{-MF[x]}$. Then if $F[x_0] < F[x] \forall x \neq x_0$, i.e. its minimum value is at x_0 ,

$$\lim_{M \rightarrow \infty} M^{-1} \log \left(\int dx e^{-MF[x]} \right) = F[x_0].$$

This works because, as M increases, the distribution that $e^{-MF[x]}$ represents becomes narrower and focuses on x_0 until the distribution has zero value at all other x .

Let us assume that N is large and take the limit over M first. We can pull the constant terms out of the integral of $Z'_N(M)$ over Ψ . Because F is positive definite, under Fubini's theorem we may switch the integrals to obtain

$$Z'_N(M) = e^{N^2 \beta L / 4 \log R^2 - \mu N L R^2} \int_{-\infty}^{\infty} \frac{d\sigma}{2\pi} \int d\Psi e^{-NF'[i\sigma]}, \quad (34)$$

where

$$F'[i\sigma] = \alpha\beta \sum_{k=1}^M \frac{1}{2} \frac{|\psi(k+1) - \psi(k)|^2}{\epsilon} + i\sigma \left(\sum_{k=1}^M |\psi(k)|^2 - MR^2 \right), \quad (35)$$

is the part of the free energy still dependent on Ψ .

The interior integral needs evaluation. Let $s = i\sigma$ and define a new partition function,

$$Z'(M) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi} \int d\Psi e^{-F'[s]}, \quad (36)$$

be that interior. Let us put F' in matrix form:

$$F'[s] = -sMR^2 + K\Psi^\dagger A\Psi + s\Psi^\dagger \Psi, \quad (37)$$

where $K = \alpha\beta/\epsilon$, and the $M \times M$ matrix A has the form

$$\begin{aligned} A_{i,i} &= 1, \\ A_{i,i+1} &= A_{i+1,i} = A_{1,M} = A_{M,1} = -\frac{1}{2}, \\ A_{i,j} &= 0 \quad \text{other } i, j. \end{aligned}$$

The integral in Equation 36 is Gaussian. We can evaluate it, knowing the eigenvalues of the matrix A . These eigenvalues have the form

$\lambda_i = 1 - \cos(2\pi(i-1)/M)$, (not related to the previous use of λ_i as strength of vorticity)[4, 26] and so

$$Z'(M) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi} e^{sMR^2} \pi^M \prod_i \frac{1}{s + K\lambda_i}. \quad (38)$$

We need to put $Z'(M)$ back into the correct form for steepest descent.

Following the example of Berlin and Kac, let $s = K(\eta - 1)$ then

$$Z'(M) = \int_{-i\infty}^{i\infty} \frac{d\eta}{2\pi} K\pi^M (\eta - 1)^{-1} e^{-M \log(K)} e^{Mf[\eta]}, \quad (39)$$

where

$$f[\eta] = KR^2(\eta - 1) - M^{-1} \sum_{i=2}^M \log(\eta - \cos(2\pi(i-1)/M)), \quad (40)$$

and $\eta \geq 1$.

We leave the $i = 1$ term out of the sum in $f[\eta]$ so that we can evaluate f

further by taking the limit on the second term,

$$\lim_{M \rightarrow \infty} M^{-1} \sum_{i=2}^M \log(\eta - \cos(2\pi(i-1)/M)) = \frac{1}{2\pi} \int_0^{2\pi} d\omega \log(\eta - \cos(\omega)),$$

which gives

$$\begin{aligned} f[\eta] &= KR^2(\eta - 1) - \frac{1}{2\pi} \int_0^{2\pi} d\omega \log(\eta - \cos(\omega)) \\ &= KR^2(\eta - 1) - \log(\eta + (\eta^2 - 1)^{\frac{1}{2}}). \end{aligned} \quad (41)$$

To apply steepest descent, we determine the saddle point $\eta = \eta_0$ where $f[\eta]$ has its minimum value, $f[\eta_0]$. Taking the derivative and setting it equal to zero,

$$\frac{\partial f}{\partial \eta} = KR^2 - \frac{1}{\sqrt{\eta^2 - 1}} = 0, \quad (42)$$

implies

$$\eta_0 = \sqrt{\frac{1}{(KR^2)^2} + 1}. \quad (43)$$

Having evaluated f , we can give an equation for the original free energy.

Letting $N\beta \rightarrow \beta'$ and $\alpha/N \rightarrow \alpha'$,

$$F[\eta_0] = -\beta' L/4 \log R^2 + \mu LR^2 + M \log(K) - Mf[\eta_0], \quad (44)$$

and evaluate it as $M \rightarrow \infty$. We will drop primes on β and α now.

The term in the limit $M \log(K)$, does not depend on R^2 , and it is an unnecessary component representing the entropy of the broken filaments in the

non-interacting case, and we drop it. Now we fill in the expressions for η_0 and

K as defined above:

$$\begin{aligned}
\lim_{M \rightarrow \infty} Mf[\eta_0] &= \lim_{M \rightarrow \infty} KR^2(\eta_0 - 1) - M \log(\eta_0 + (\eta_0^2 - 1)^{\frac{1}{2}}) \\
&= \lim_{M \rightarrow \infty} MKR^2 \left(\sqrt{\frac{1}{(KR^2)^2} + 1} - 1 \right) - M \log \left(\sqrt{\frac{1}{(KR^2)^2} + 1} + \frac{1}{KR^2} \right) \\
&= \lim_{M \rightarrow \infty} M \frac{\alpha\beta M}{L} R^2 \left(\sqrt{\frac{1}{(\frac{\alpha\beta M}{L} R^2)^2} + 1} - 1 \right) - M \log \left(\sqrt{\frac{1}{(\frac{\alpha\beta M}{L} R^2)^2} + 1} + \frac{1}{\frac{\alpha\beta M}{L} R^2} \right)
\end{aligned} \tag{45}$$

which, because it is an energy for a filament, ought to be finite. The first term is the energy of the filament, E_{fil} , and the second, the entropy, S_{fil} , and each by itself is finite, so we take each limit separately.

The energy:

$$E_{fil} = \lim_{M \rightarrow \infty} M \frac{\alpha\beta M}{L} R^2 \left(\sqrt{\frac{L^2}{(\alpha\beta M R^2)^2} + 1} - 1 \right) = \frac{L}{2\alpha\beta R^2} \tag{46}$$

The entropy:

$$S_{fil} = \lim_{M \rightarrow \infty} M \log \left(\sqrt{\frac{1}{(\frac{\alpha\beta M}{L} R^2)^2} + 1} + \frac{1}{\frac{\alpha\beta M}{L} R^2} \right) = \frac{L}{\alpha\beta R^2} \tag{47}$$

These two results imply that

$$\lim_{M \rightarrow \infty} Mf[\eta_0] = -\frac{L}{2\alpha\beta R^2}$$

and

$$F[\eta_0] = L\mu R^2 - \beta L/4 \log R^2 + \frac{L}{2\alpha\beta R^2}. \tag{48}$$

We minimise with respect to R^2 ,

$$\frac{\partial F}{\partial R^2} = L\mu - \frac{\beta L}{4R^2} - \frac{L}{2\alpha\beta R^4} = 0, \tag{49}$$

and solve for R^2 ,

$$\begin{aligned} R^2 &= \frac{\beta/4 \pm \sqrt{(\beta/4)^2 + 4\mu \frac{1}{2\alpha\beta}}}{2\mu} \\ &= \frac{\beta^2\alpha + \sqrt{\beta^4\alpha^2 + 32\alpha\beta\mu}}{8\alpha\beta\mu}, \end{aligned} \tag{50}$$

where we take the “plus” solution as giving physical results.